ADMISSIBILITY OF SEMIGROUP STRUCTURES ON CONTINUA

BY

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The purpose of this note is to investigate the structure of certain compact connected semigroups S which satisfy $S^2 = S$. If X is any continuum then a continuous associative multiplication may be introduced by

(i) $xy = x$ all $x, y \in X$

or

(ii) $xy = y$ all $x, y \in X$.

We shall give certain instances in which, due to the topological structure of S, the multiplication *must* be of this trivial kind. The central ideas of the paper spring in one way or another from dense connected sets. For example, S being a continuum (S will always denote a Hausdorff topological semigroup, or mob), $S^2 = S$ if and only if each dense ideal is connected. We will show, for example, (Theorem 6) that if S is irreducible between its minimal ideal K and some other set and if $S^2 = S$ then S must have a unit, S is commutative, and S/K (in the metric case) is an arc. In a related result (Theorem 5) it will be proved that maximal ideals are composants. Because of the relative novelty of the subject and the present lack of structure theorems we go into considerable detail in considering some examples.

We follow the terminology of [1, p. 14]. In particular K denotes the minimal ideal of S, if such exists, otherwise $K = \square$, and E denotes the set of idempotents. If S is compact it is known (see e.g. [14]) that K and E are nonempty and that K is completely simple. If S is connected and K is nonempty, then K is connected. For $a \in S$, $J(a) = a \cup Sa \cup aS \cup SaS$ (the ideal generated by a). A clan is a compact connected mob with unit. We recall a result of [8]: if S is compact and $S^2 = S$, then $S = SES$. This is equivalent to: if S is compact, $S^2 = S$, and M is a proper ideal of S then there is an idempotent outside M.

We first state a topological extension [19; 20] of a theorem of Rees-Suschkewitsch [13].

THEOREM 1. *Let S have a compact kernel ($=$minimal ideal) K, and let $e \in K$. Form $eSe \times (eS \cap E) \times (Se \cap E)$ with the multiplication $(t, e, f)(t', e', f') = (tef't, e', f)$ and define $\phi: eSe \times (eS \cap E) \times (Se \cap E) \to K$ by $\phi(t, e, f) = fet$. Then $eSe \times (eS \cap E) \times (Se \cap E)$ is a compact mob, and ϕ is a topological isomorphism*

onto. Further, if $\eta\colon K{\to}K$ *is defined by "$\eta(t)$ is the idempotent in the group containing t," then* $r\colon S{\to}K$ *defined by* $r(x) = x\eta(ex)$ *is a retraction of S onto K.*

Our main use of Theorem 1 will be in the form of the following corollary:

COROLLARY 1. *Let S be a continuum, and suppose K is not the cartesian product of two nondegenerate continua. Then either K is a group or the multiplication in K is of type* (i) *or* (ii).

Proof. From Theorem 1 we know K is topologically isomorphic with $eSe\times(eS\cap E)\times(Se\cap E)$, and each of the factors is a continuum since S is. Hence at least two of eSe, $eS\cap E$, and $Se\cap E$ must consist of single elements. If $eS\cap E$ and $Se\cap E$ are single elements (namely e), then K is isomorphic with eSe, hence is a group [1]. If eSe and $eS\cap E$ are both the element e, then the multiplication is of type (i). If eSe and $Se\cap E$ are both e, then the multiplication is of type (ii).

The following theorem extends a result of Faucett [4, Theorem 1.4].

THEOREM 2. *Let S be compact and connected. Then* $S^2 = S$ *if and only if each dense left (right) ideal containing K is connected (also, if and only if, each dense ideal is connected).*

Proof. Suppose $S^2 = S$ and let L be a dense left ideal containing K. Then $SL = S(L\cup K) = \cup\{Sa\,|\,a\in L\}\cup K$ and each Sa meets the connected set K, so SL is connected. Now $SL\subset L\subset S$ and $(SL)^* = S^2 = S$ (stars denote closure) so L is connected. Conversely, suppose each dense left ideal containing K is connected and $S^2\neq S$. Let $a\in S\backslash S^2$ and let V be an open set about a with $V^*\cap S^2 = \square$. Then $S\backslash F(V)$ is a dense two-sided ideal and is not connected, a contradiction.

LEMMA 1. *Let S be connected, let K exist and let S = ES. Then each left ideal containing K is connected.*

Proof. Let L be a left ideal containing K. Since $S = ES$ it follows that $L = SL$. Then $L = SL = S(L\cup K) = \cup\{Sa\,|\,a\in L\}\cup K$ and each Sa meets the connected set K, so L is connected.

We say that a continuum has property ($*_n$) if each dense connected open subset omits at most n points.

LEMMA 2. *Let S be a continuum satisfying* ($*_2$), *and with* $S^2 = S$. *Then either S has a left or right unit, or for some e,* $f\in E$, $S = Se\cup Sf = eS\cup fS$.

Proof. It is known that each maximal proper (left, right, 2-sided) ideal is open and dense [9, p. 681–682]. Note that S has at most two maximal proper ideals (m.p.i.): for if M_1, M_2, M_3 are m.p.i. then $M_1M_2M_3\subset M_1\cap M_2\cap M_3$. Now $M_1M_2M_3$ is dense and connected by continuity of multiplication, hence $M_1\cap M_2\cap M_3$ is dense. It is open since each of M_1, M_2, M_3 is open, and finally

it is connected since it lies between a connected set and its closure. Hence $M_1 \cap M_2 \cap M_3$ is dense, connected, open, and omits at least three points.
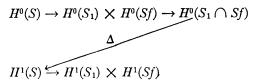
CASE 1. *M is the unique m.p.i.* Then M omits at most two elements, and at least one of these is an idempotent, say $M \subset S \backslash e$. Now Se and eS may be assumed to be proper since otherwise S would have a left or right unit. Hence $Se \subset L$, a maximal proper left ideal and $eS \subset R$, a maximal proper right ideal. Let $a \in S \backslash L$ and $b \in S \backslash R$. Now $RML \subset R \cap M \cap L$ so $R \cap M \cap L$ is dense, connected, open and omits e, a, b. Hence $a = b$, $R = L$, and L is an ideal. Since M is unique, $e \in L \subset M$, a contradiction.

CASE 2. M_1, M_2 *are distinct m.p.i.* Then $M_1 = S \backslash e$, $M_2 = S \backslash f$ where e, $f \in E$. If $Se \cup Sf$ is proper, then $Se \cup Sf \subset L$, a maximal proper left ideal. Let $a \in S \backslash L$. Then $M_1 M_2 L \subset M_1 \cap M_2 \cap L$ and this omits the three points e, a, f. We conclude that $Se \cup Sf = S = eS \cup fS$, and the proof is complete.

In what follows, we use the Alexander-Kolmogoroff cohomology groups based on some fixed but arbitrary nonzero Abelian group for coefficients.

LEMMA 3. *Let S be a continuum with either left or right unit, or with $S = E_1 S$* $= SE_2$ *where $E_1 \cup E_2 \subset E$ and one of E_1, E_2 is finite. Then $H^1(S, K) = 0$.*

Proof. Note that $H^1(S, K)$ is isomorphic with $H^1(S/K)$ where S/K is the Rees quotient of S by K [16]. Since S/K has a zero and satisfies the hypotheses of the lemma, it suffices to consider the case when S has a zero. It now remains to show $H^1(S) = 0$. If S has either a left or right unit, the result is known [16]. Suppose E_2 is finite and assume inductively that $H^1(S) = 0$ if card $E_2 \leqq k$. Suppose $S = E_1 S = SE_2$ with $E_2 = \{e_1, e_2, \cdots, e_k, f\}$, and let $S_1 = \cup_{i=1}^{k} Se_i$. Consider the Mayer-Vietoris exact sequence [3] (using *reduced* groups in dimension 0):

$$H^0(S) \to H^0(S_1) \times H^0(Sf) \to H^0(S_1 \cap Sf)$$
$$\Delta$$
$$H^1(S) \overset{\leftarrow}{\to} H^1(S_1) \times H^1(Sf)$$

Now S_1 and Sf are connected since S has a zero, so $H^0(S_1) = 0$ and $H^0(Sf) = 0$. Also $H^1(S_1) = 0$ by the induction hypothesis, and it is known (since S has a zero) that $H^1(Sf) = 0$ [16]. It follows that Δ is an isomorphism onto, and it remains to show that $H^0(S_1 \cap Sf) = 0$, i.e. that $S_1 \cap Sf$ is connected. Since S has a zero and $S = E_1 S$, this is immediate from Lemma 1, and the proof is complete.

We say that a continuum X has *property* N if for any proper subcontinuum $P \subset X$, $H^1(X, P) \neq 0$.

THEOREM 3. *Let S be a continuum satisfying $(*_2)$, having property N, and with $S^2 = S$. Then $S = K$, and either S is a group or the multiplication is of type* (i) *or* (ii).

Proof. We conclude from Lemmas 2 and 3 that $H^1(S, K) = 0$. Hence if $K < S$, then S fails to have property N. Therefore $S = K$, and since S cannot be the cartesian product of two nondegenerate continua by ($*_2$), the conclusion follows from Corollary 1.

Some examples covered by Theorem 3 are: S^1, two tangent S^1's, S^1 together with a diameter, two disjoint S^1's connected by a line segment, and variations of these in which arcs are replaced by "long" arcs (based on the ordinal numbers filled in by arcs). It is likely that the only mob satisfying the hypotheses of Theorem 3, and which admits a group structure is S^1. At this writing it is unknown whether Theorem 3 holds with ($*_2$) replaced by ($*_n$), and in particular by ($*_3$).

DEFINITION. Let X be a continuum, $p \in X$, and define $T(p)$ to be the intersection of the sets $(X \backslash A)^*$ where A runs through all continua in X not containing p. As stated in [7], $T(p)$ is a continuum if X is metric. It can be seen that this holds without the assumption that X is metric.

THEOREM 4. *If S is a clan, if I is an open or closed ideal, and if $p \in S$, then $T(p) \cap I \neq \square$ implies $p \in I$.*

Proof. Let I be closed, $p \in S \backslash I$, $x \in T(p) \cap I$. Let U be an open set about p with $U^* \cap I = \square$. Now $J_0(S \backslash U^*)$, the largest ideal of S contained in $S \backslash U^*$, is an open connected ideal [9] containing I. Then $J_0(S \backslash U^*)^*$ is a continuum containing x in its interior which does not contain p. Hence $x \in S \backslash T(p)$, a contradiction. Now let I be open, $p \notin I$, $x \in T(p) \cap I$. Let U be an open set about x with $U^* \subset I$. Note that SU^*S is an ideal and hence is connected since S has a unit. Then SU^*S is a continuum containing x in its interior which does not contain p. Hence $x \in S \backslash T(p)$, a contradiction.

COROLLARY. *If S is a clan and if $p \in H_u$ (the maximal subgroup containing the unit), then $T(p) \subset H_u$.*

Proof. Note that $S \backslash H_u$ is a maximal proper ideal. If $x \in T(p) \backslash H_u$, then $S \times S$ is a closed ideal containing x, so by the theorem, $p \in S \times S$. But $p \in H_u$ implies $H_u \subset S \times S \subset S \backslash H_u$, a contradiction.

EXAMPLE 1. Let A be the graph of $\sin x^{-1}$, $0 < x \leq 1$ together with the limit line, let $B = \{(2 - x, y) | (x, y) \in A\}$, and let $X = A \cup B$. Denote the limit lines by L and R. We show that if X admits the structure of a mob S equal to its square then the multiplication must be of type (i) or (ii). Suppose $S^2 = S$, $K < S$ and let J be a maximal proper ideal. Then there is an idempotent e outside J, and since J is open, dense, and connected by Theorem 2, we have $e \in L \cup R$. We show now that if $K \cap L \neq \square$, then $K \cup L$ is an ideal. If $L \subset K$ then the claim is clear. Hence we may assume $K \subset L$. Now $SL = LL \cup [S \backslash (L \cup R)] \cdot L \cup RL$ and each of these sets is arcwise connected and meets $K \subset L$. Hence each set is contained in L since no arcwise connected set can meet both L and its complement. In a similar way $L \cup K (= L)$ is a right

ideal, so $L \cup K$ is an ideal. In a similar way we can show that if $K \cap R \neq \square$ then $K \cup R$ is an ideal. Since K is proper, either $K \cap L = \square$ or $K \cap R = \square$. Also either (1) $K \cap L = \square$ and $e \in L$ or (2) $K \cap R = \square$ and $e \in R$. Suppose (1) holds. Let f be any idempotent in L. Then fSf is a continuum containing f and a point of K (which is outside L). Hence $L \subset fSf$, so f is a unit for L. It follows that e is the unique idempotent of L. Since L is arcwise connected, so is L^2, and since $e \in L$ we conclude $L^2 \subset L$. It follows that L is a group, but L is not homogeneous, a contradiction. Under (2) we reach a similar contradiction. Hence $K = S$ and the conclusion follows from the Corollary to Theorem 1.

EXAMPLE 2. [11, p. 93]. Let p be the origin of the plane, $a = (1, 0)$, $b = (-1, 0)$, $b_n = (-1, -1/n)$, $c_n = ((n+1)/n, -1/n)$, $e_n = ((n+1)/n, 1/n)$. Let L be the line segment from b to a, let x_n be the arc from b_n to p, $x_n = [b_n, c_n] \cup [c_n, e_n] \cup [e_n, p]$. Let $S = L \cup x_1 \cup x_2 \cup \cdots$. We show that S does not admit the structure of a mob with unit. Suppose the contrary. We first show $K \cap L = \square$. Suppose not; then if x is a point of L to the right of p, we have $T(x) = L$, so $T(x) \cap K \neq \square$ and $x \in K$. Note that no subcontinuum of S is a cartesian product of two nondegenerate subcontinua, so by the Corollary to Theorem 1 we may assume K consists of left zeroes, i.e. each element of K is a minimal right ideal. Let V be an open set about x, let $R(V)$ be the largest right ideal of S contained in V. Then $R(V)$ is open [9] and $R(V) \cup K$ is connected (by the dual to Lemma 1). We conclude that K contains all but a finite number of line segments converging to $[p, a]$, (the segment from p to a). Now the unit u is a non cutpoint, and we distinguish two cases. Suppose $u \in S \setminus L$; pick sequences $\{w_n\}$ and $\{z_n\}$ in k with $\{z_n\} \to p$ and $\{w_n\} \to x$. Then $u z_n = z_n$, $p z_n = p$ for each positive integer n so by arcwise connectedness there is a sequence $\{t_n\}$ in $[p, u]$ with $t_n z_n = w_n$. Now $\{t_n\}$ clusters at $t_0 \in [p, u]$ so $\{t_n z_n\} = \{w_n\}$ clusters at $t_0 p$ and converges to x, hence $t_0 p = x$. We show next that $[p, u]$ is a mob, which will furnish a contradiction. Let q be that point of K in $[p, u]$ which is closest to u. By a result of Faucett [4], $[q, u]$ is a mob. Let r be that point of $[p, q]$ which is closest to q and satisfies $p \in [u, q]r$, so that $[u, r]^2 = [u, p]$. Using the fact that $[q, r]$ consists of left zeroes we see that $[u, r]^3 = [u, r]^2$. Hence $\cup \{ [u, r]^n \mid n \geq 1 \} = [p, u]$ is a mob. Hence we may suppose $u \in L$; let $\{x_n\}$ be a sequence of endpoints converging to u. Then $u x_n = x_n$ and $p x_n = p$ for each n, so that, by arcwise connectedness, there is a sequence $\{t_n\}$ in $[u, p]$ with $t_n x_n = w_n$. Then $\{t_n\}$ clusters at $t_0 \in [u, p]$ so $\{t_n x_n\} = \{w_n\}$ clusters at $t_0 u$ and converges to x, hence $t_0 = x$, a contradiction. Thus $K \cap L = \square$ so that K is an interval. We shrink K to a point to get the Rees quotient [13], and now have a zero (0). Let y denote the endpoint of the line on which 0 lies. Then $[y, p)$ is an open set about 0, so $J = J_0([y, p))$, the largest ideal of S contained in $[a, p)$, is an open ideal with $J^* \cap L = p$. Hence $T(x) \cap J^* \neq \square$, so that $x \in J^*$, the contradiction which completes the argument.

Recall that A is an isolated arc in the space X if A is an arc and $A \cap (X - A)^*$ is one endpoint of A.

REMARK. *Let X be a space containing an isolated arc. Then X admits the structure of a mob S with left unit and with $S \neq K$.*

Proof. Let A be an isolated arc in X, $B = (X-A)^*$, $p = A \cap B$. We define the multiplication in B to be of type (ii). Let the multiplication in A be that of the usual unit interval with p playing the role of zero. Finally, for $x \in A$, $y \in B$, let $xy = y$, $yx = p$.

The construction can perhaps be seen more easily in the following way: using the multiplications in A and B given above, let $A \times B$ have coordinate-wise multiplication. Then $A \cup B$ is homeomorphic with the mob $(A \times p) \cup (p \times B)$.

We now take up some results of a preliminary nature involving a mob S which is a continuum irreducible between two subsets A and B and with $S^2 = S$. Recall that S is irreducible between A and B if for any subcontinuum C with $C \cap A \neq \square \neq C \cap B$ we have $C = S$. For a compact Hausdorff space X, $cdX \leq n$ iff the natural map $H^n(X) \to H^n(A)$ is onto for each closed set $A \subset X$ [2]. (Here cdX is based on a fixed, but arbitrary, coefficient group.)

THEOREM 5. *Let S be a continuum with neither left nor right unit such that $S^2 = S$ and S is irreducible between two subsets A and B. Then*

(1) *$A \subset H_e$, $B \subset H_f$ for some e, $f \in E \setminus K$, and $S = eSe \cup fSf$.*
(2) *There are exactly two maximal proper ideals, $S \setminus H_e$ and $S \setminus H_f$.*
(3) *H_e and H_f are connected.*
(4) *S has exactly three composants, S, $S \setminus H_e$ and $S \setminus H_f$.*
(5) *If $cdS = 1$, then $A = H_e = e$, $B = H_f = f$.*

Proof of (1). Let $p \in A$, $q \in B$; since $S^2 = S$, $p \in xS$ and $q \in Sy$ for suitable x, $y \in S$. Hence by the irreducibility $S = xS \cup Sy$ and $xS = x^2S \cup xSy$, so that $S = x^2S \cup xSy \cup Sy = \cdots = x^nS \cup Sy$ for each positive integer n. Therefore $S = e'S \cup Sy$ where $e' \in E \cap \Gamma(x)$ [8, Theorem 2]. By a similar argument $S = e'S \cup Sf'$ where $f' \in \Gamma(y)$. If $p \notin e'S$ then $p \in Sf' \subset Sy$ so that p, $q \in Sy$ and $Sy = S$; it follows that S has a right unit. Hence $p \in e'S$ and similarly $q \in Sf'$, $p \in Sp$, $q \in qS$, and $S = Sp \cup qS$. Now $S = Se \cup fS$ for suitable idempotents e, f, and $p \in Se$, $q \in fS$, $p \in pS$, $q \in Sq$. Hence $S = pS \cup Sq$, so $S = eS \cup Sf$. If $p \notin eS$, then p, $q \in Sf$, $Sf = S$, and f is a right unit for S. Hence $p \in eS$ and similarly $q \in Sf$. Hence $p \in eS \cap Se = eSe$ and $q \in fS \cap Sf = fSf$, so by the irreducibility $S = eSe \cup fSf$. Since $e \in pS \cap Sp$ it follows that $p \in H_e$ and in a similar way $q \in H_f$.

We now show e, $f \in S \setminus K$. If $e \in K$, then either $(eSe)^0 = \square$ or $eSe = K$ [15, Theorem 4]. The first of these is impossible since $e \notin fSf$. But if $eSe = K$, then K is a group [14], hence $K \subset fSf$. Then $e \in K \subset fSf$, the contradiction which completes the proof of (1).

Proof of (2). Note that any maximal proper ideal M misses either A or B, otherwise for $a \in A \cap M$, $b \in B \cap M$ we have $J(a) \cup J(b) \subset M < S$. But by Lemma 1, $J(a) \cup J(b)$ is a continuum which meets both A and B, so that S

is not irreducible. It follows that there are at most two maximal proper ideals and we now show that $S\backslash H_e$ and $S\backslash H_f$ are distinct maximal proper ideals. The distinctness is clear and it suffices to show that $S\backslash H_e$ is an ideal. If not, then $S\backslash H_e$ fails, say, to be a left ideal, so for some $x\in S$ and $y\in S\backslash H_e$, $xy\in H_e$. We conclude that $e\in Sy$; further since $S = eSe\cup fSf$ and S has neither a left nor a right unit it follows that $y\in eSe$. Hence $e = zy$ for some $z\in S$ and we may assume z, $y\in eSe$. Now $(yz)(yz) = yz$. So by the complete simplicity of $SeS/(SeS\cap J_0(S\backslash e))$ [6, Corollary 4] either $yz = e$ or $yz\in J_0(S\backslash e)$. If $yz\in J_0(S\backslash e)$ then $e = (zy) = z(yz)y\in J_0(S\backslash e)$, a contradiction. Hence $yz = e$ and $y\in H_e$. This contradiction completes the proof.

Proof of (3). We establish the following more general result. The arguments used are similar to those in [17, Lemma 3]. *Let S be a continuum, $S = ESE$, $K\neq S$, S irreducible about A, and let J be a maximal proper ideal. Then A^* meets each component of $S\backslash J$.*

Proof. Let C be a component of $S\backslash J$, suppose $A^*\cap C = \square$, and let U be an open set about C with $F(U)\cap(S\backslash J) = \square$, and with $A^*\cap U^* = \square$. Then $F = F(U)\subset J$ so $SFS\subset J$. Since J is open, there is an open set V about F with $V\cap K = \square$ and $SV^*S\subset J$. Now $S\backslash F = (S\backslash U^*)\cup U$ so $S\backslash SV^*S = [S\backslash(U^*\cup SV^*S)]\cup[U\backslash SV^*S]$ and these are separated. (Note that $F\subset SV^*S$.) If $W = U\backslash SV^*S$ then $S\backslash W$ is connected, $W\subset U$ and $C\subset W$. Hence $A\cap W = \square$ and $(S\backslash W)^*$ is a proper continuum containing A, violating the irreducibility.

Proof of (4). We show that C_e, the composant of S containing e, is $S\backslash H_f$. Let $x\in S\backslash H_f$; then $J(x)\cup K\cup eSe$ is a proper continuum containing e and x so that $x\in C_e$. Now let $x\in C_e$, so that x, $e\in P$, a proper subcontinuum of S. Suppose $x\in H_f$; then $H_e\cup P\cup H_f$ is a continuum containing p, q, so $H_e\cup P\cup H_f = S$. But H_e has no inner points since maximal proper ideals are dense, and similarly for H_f. Hence $P = S$, a contradiction.

Proof of (5). We first form the Rees quotient S/K, i.e. we shrink K to a point. If $f\colon S\to S/K$ is the natural map, then S/K is a continuum irreducible between $f(A)$ and $f(B)$ and has a zero. Hence we may assume S has a zero. If $H_e\neq e$, then H_e is a compact connected one-dimensional group, so $H^1(H_e)\neq 0$ and $H^1(eSe)\neq 0$ (real coefficients). But eSe is now a clan with zero, and it is known [16, p. 48] that $H^n(eSe) = 0$ (all $n\geq 0$), a contradiction.

Recall that a subset C of a continuum X is a C-set if for any subcontinuum A of X with $A\cap C\neq\square$ we have either $A\subset C$ or $C\subset A$.

THEOREM 6. *Let S be a continuum such that $S^2 = S$ and such that S is irreducible between K and A, where $A\subset S$. Then*

S has a unit, S is commutative, and S/K is irreducibly connected between zero and unit.

The proof will proceed in several short steps, as follows:

(1) *S has a unit u, $A\subset H_u$, and H_u is the set of all x such that S is irreducible between K and x.*

(2) K *is a group, each closed left ideal is principal,* $xS = Sx$ *for each* $x \in S$, *and* $x, y \in S$ *imply* $Sx \subset Sy$ *or* $Sy \subset Sx$.

(3) *The subcontinua of* S *meeting both* K *and* $S \backslash K$ *coincide with the principal ideals* $\neq K$, $I(p)$ *is open for each* $p \in S \backslash K$, *and each set* J_a, $a \in S$, *is a C-set in* Sa.

(4) $J_a = a$ *for each* $a \in S \backslash K$, *so* S/K *is irreducibly connected between zero and unit.*

(5) S *is commutative.*

Proof of (1). Let $a \in A$; since $S^2 = S$, $a \in Sx$ for some $x \in S$. Now Sx is a continuum which meets K and contains a so that $S = Sx$. Hence $S = Se$ where $e \in \Gamma(x) \cap E$, and e is a right unit. Dually S has a left unit so that S has a unit. Note that by the result stated in the proof of Theorem 5, Part (3), H_u is connected. We show next that H_u is the set of all x such that S is irreducible between K and x. Suppose S is irreducible between K and x; then $Sx = S$ so $x \in H_u$. Conversely if $x \in H_u$, then let P be a continuum with $x \in P$ and $P \cap K \neq \square$. If $a \in A$, then $ax^{-1}P = S$, and by [18, Theorem 1] we conclude that $P = S$.

The following lemma will be useful in the remaining proofs.

LEMMA 4. *Let* S *satisfy the hypotheses above, and let* $L \subset S$ *be a closed left ideal containing* K. *Then* $x \in S - L$ *implies* $Sx \supset L$.

Proof. Let U be an open set about x with $U^* \cap L = \square$. Let $P = L_0(S \backslash U^*)$, the largest left ideal of S contained in $S \backslash U^*$. Since S has a unit by (1), P is connected by Lemma 1, and P is known to be open [9]. Since $K \subset P$, it follows from [10, p. 134] that $(S \backslash P^*)^*$ is a continuum (which misses L). Hence $Sx \cup (S \backslash P^*)^*$ is a continuum which meets K and contains u, so $Sx \cup (S \backslash P^*)^* = S$ and $L \subset Sx$.

Proof of (2). We show first that K is a group. If we take $L = K$ in Lemma 4, and let $y \in F(K)$, it follows from continuity of multiplication that $Sy = K = yS$. Hence K is a minimal left and right ideal, so K is a group [1]. Since K is a group, each left ideal must contain K, and the hypotheses of Lemma 4 can be relaxed accordingly. By an argument similar to that just given for K, we see that each closed left ideal is principal. We show next that $Sx = xS$ for each $x \in S$. To show that $Sx \subset xS$ let $y \in Sx$ and suppose $y \in S \backslash xS$. By the dual of Lemma 4, $xS \subset yS \subset SxS$. But xS is maximal among the principal right ideals generated by elements of SxS [8, p. 399], so $xS = yS$ and $y \in xS$, a contradiction. In a similar way, $xS \subset Sx$, so $Sx = xS$. Finally let $x, y \in S$ and suppose $Sx \not\subset Sy$. Note that $x \in S \backslash Sy$ so by Lemma 4, $Sy \subset Sx$ and (2) is complete.

Proof of (3). Let P be a continuum of S which meets both K and $S \backslash K$. Then $P \subset SP$, and since each closed left ideal is principal, $SP = Sx$ for some $x \in S$. Also $x \in SP$, so $x \in Sp$ for some $p \in P$, and $SP = Sx = Sp$ (note that $p \in S \backslash K$). Hence $P \subset Sp$, and we now show $Sp \subset P$. Let $J_p = \{x \mid SxS = SpS\}$,

and let $I(p) = J(p)\backslash J_p$. It is clear that $I(p)$ is an ideal of S, and we first show $I(p)$ is open. Let $x \in I(p)$; by (3), $SxS = Sx = xS$, so $Sx < Sp$. Let V be an open set about Sx with $p \in S\backslash V^*$. Let L be the largest left ideal of S contained in V; then L is open [9] and $p \in S\backslash L^*$. Hence by Lemma 4, $x \in L \subset L^* \subset Sp$, so $I(p)$ is open. Now let $y \in I(p)$, and let W be an open set about y with $W \subset I(p)$ and $W^* \cap J_p = \square$. Then $(S\backslash SW^*)^* \cup P$ is a continuum which meets both A and K, so that $(S\backslash SW^*)^* = S$ and it follows that $x \in P$. Hence $I(p) \subset P$ and $Sp = J(p) = I(p)^* \subset P$.

Finally we show that each J_a, $a \in S$, is a C-set in Sa. Let Q be a subcontinuum of SaS with $Q \cap J_a \neq \square \neq Q \cap I(a)$, and let $x \in Q \cap I(a)$. Then $SxS \cup Q$ is an ideal by the above results, hence $Sx \cup Q = J(a)$. Since $Sx \subset I(a) \subset S\backslash J_a$, $J_a \subset Q$ and J_a is a C-set. It is known [15, p. 639] that a C-set is connected and has no inner point.

Proof of (4). Let $a \in S$ and we will show $J_a = a$. We distinguish two cases, according as J_a is a group or not. We note that J_a is a group if and only if $J_a \cap E \neq \square$. This follows from the complete simplicity of $J(a)/I(a)$ [6] and the fact that $Sx = xS$ for each $x \in S$. Suppose J_a is not a group; let Se be a minimal member of $[Sf | a \in Sf, f \in E]$. Then in the sense of the relation $a \leq b \leftrightarrow a \in Sb$, e is *the* smallest idempotent above a (the uniqueness is a consequence of part (2)). It follows that $V = Se\backslash Sa$ is an open set relative to Se containing e such that $V \cap E = \{e\}$. By a theorem of Mostert and Shields [12] an arc A can be started at e such that $A \subset Se$ and $A \cap J_e = \{e\}$. Since J_e is a C-set in Se, we conclude that $J_e = e$. Now $Sb = [x | x \leq b]$ is closed for each $b \in S$. Hence [21] there is a smallest ($\leq$) element z with $za \in J_a$. We now show that $J_z \cdot a = J_a$. If $y \in J_z$ and $ya \in I(a)$, then $za \in Sza = Sya \subset I(a)$, a contradiction. Now let $b \in J_a$ and suppose that $b \notin J_z \cdot a$. Note that $b \notin I(z)a$, otherwise (using the minimality of z) $za \in zSb \subset SI(z) \cdot a \subset I(a)$, contrary to fact. Hence $b \notin [I(z) \cup J_z] \cdot a = J(z) \cdot a = Sza = Sa$, a contradiction. Hence $J_z \cdot a = J_a$, so $a = ta$ for some $t \in J_z$, and $a = ta = t^2a = \cdots = t^na$ for each positive integer n. Therefore $a = fa$ for $f \in E \cap \Gamma(t)$, and $a \in fS$. We conclude from the minimality of e that $f = e$, $t \in H_e$, and $J_z = H_e = \{e\}$. Hence $J_a = J_z \cdot a = ea = a$.

Now suppose J_a is a group, $J_a = J_e = He$ with $e^2 = e$. Suppose $J_e \neq e$, and consider Se. Since H_e is not a D-chain in Se, [4, Theorem 2.2], there is an open set V_1 (relative to Se) about H_e such that V_1 contains no cutpoint of Se. Let V be an open set (relative to S_e) about H_e with $V^* \subset V_1$. If $V \cap (E\backslash e) = \square$, then an arc (in Se) can be started at e and as in the first part of the proof we conclude $J_e = e$. Hence we may assume $V \cap (E\backslash e) \neq \square$. We show next that $V^* \cap E$ is a compact order dense ($\leq$)-chain. It is clearly a compact chain. To show $V^* \cap E$ is order-dense, suppose $f, g \in V^* \cap E$ and g "covers" f, that is, g is the least element with $f \in Sg$ and $g \neq f$. Then there is W open (relative to Sg) about g with $W \subset V$ and $f \in S\backslash W$. Now $W \cap (E\backslash g) = \square$, so an arc A can be started in Sg, with $A \cap J_g = \{g\}$. Since J_g is a C-set we conclude that $J_g = \{g\}$, hence g is a cutpoint of S since $I(g)$ is open. But V contains no cut-

points of S, a contradiction. It is clear that the topology induced by $\leq$ on E coincides with the relative topology of E. Hence $V^*\cap E$ is an order dense compact chain and it follows that $V^*\cap E$ is a continuum. Since J_e is a C-set and $V^*\cap E\cap J_e = \{e\}$, we conclude that $J_e = \{e\}$. Therefore $J_a = \{a\}$ for each $a\in S\backslash K$, and each $a\in S\backslash K$ is a cutpoint. It is now immediate that S/K is irreducibly connected between a zero and a unit.

Proof of (5). We show now that S is commutative. By a result of Faucett [5, Lemma 5], S/K is commutative. If K consists of a single element, then $S = S/K$, so that S is commutative. Now suppose that K is not a single element and we will show that $S\backslash K$ is a mob. If not, then for some $x, y\in S\backslash K$ we have $xy\in K$. Note that by Part (3) of the theorem, K is a C-set, and hence contains no inner point. Let $[y, u]$ denote the "segment" from y to the unit. Then $x\cdot[y, u]$ is a locally connected continuum which meets K and contains x, so $K\subset x\cdot[y, u]$. But since K is a nondegenerate C-set, $x\cdot[y, u]$ is not locally connected at any element of K, a contradiction. Hence $S\backslash K$ is a mob, and is commutative since S/K is. We conclude that $(S\backslash K)^* = S$, so that S is commutative.

COROLLARY 1. *If S is a metric indecomposable continuum with $S^2 = S$, then either S is a group or the multiplication is of type* (i) *or* (ii).

Proof. If K is proper, then S is irreducible between K and some point not in K, so that by the theorem, S has a unit. Hence by [9, Theorem 3] S is a group and $S = K$, a contradiction. We now have $S = K$ and the result follows from the corollary to Theorem 1.

The following example indicates the complexities that may occur even under what may seem to be the strong hypotheses of the theorem. Let G be a compact group which contains a dense one-parameter semigroup $f(R_+)$ (here R_+ denotes the non-negative additive reals). Define $g: R_+\to R$ by $g(t) = \exp(-t)$. If $h = f\times g$ then h takes R_+ isomorphically into $G\times(0, 1)$ and $h(R_+)^*$ is a clan whose minimal ideal is the group $G\times\{0\}$ and which is irreducible from $G\times\{0\}$ to $(e, 1)$, e the unit of G. We may describe $h(R_+)^*$ by saying it is the group G and an infinite half ray spiraling down upon it.

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